Stuart Welsh

What's the problem?

The Complete Mathematics Conference

@maths180

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What's the problem?

The Complete Mathematics Conference

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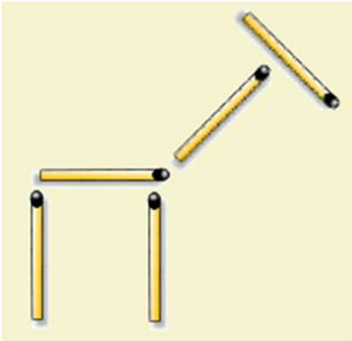
This is a presentation about solving problems in mathematics. It's split into three parts.
In the first part, I will talk about why problem solving matters, explore the features of a good problem, offer a definition of what it means to solve a problem and explain why the difficulty of a problem is as much about the solver as it is the problem itself.
In part two, we'll tackle the question of whether we can actually teach problem solving. I'll give a brief history of research on problem solving, introduce some common problem-solving frameworks and discuss the differences in approaches taken by experts and novices when solving problems.
In the third and final part of this presentation, I will make some suggestions as to what can be done in schools and classrooms to help pupils become better at solving problems.

This short presentation is really an overview of a series of CPD sessions. If you like what you hear and think your department would enjoy some bespoke problem solving CPD, then get in touch on email or twitter.

Use 5 matchsticks (or algebra tiles) to construct this "donkey".



By moving just **one** matchstick, turn the donkey.

http://www.learning-tree.org.uk/phpBB3/viewforum.php?f=1

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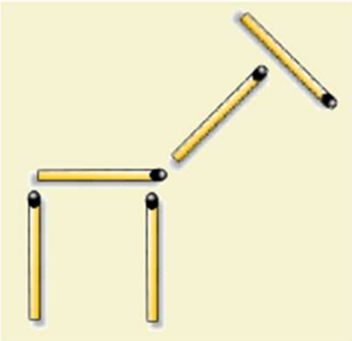
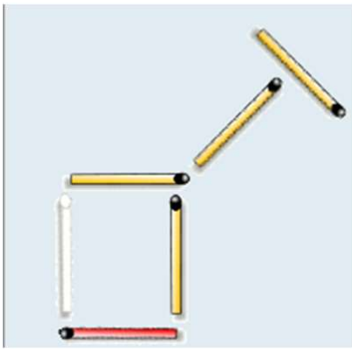
Try this task.

Use 5 matchsticks (or algebra tiles) to construct this “donkey”.





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How did you feel when faced with this task? – Excited, deflated, curious, confused, bored, inadequate, superior?

How did you approach it? – draw on previous experience? Use a systematic strategy? A moment of insight?

How did you feel when you solved it (or saw the solution)? – Relieved? Cheated? Justified? Smart? Not smart?

Think for a moment about all the emotions that are at play here and how different the experience will be for different solvers.

This task is a better referred to as a “puzzle” rather than a problem as it requires little or no prior knowledge to solve it.
However, what it does have in common with a good problem is the need for a little a “flash” of inspiration to solve it.
It is worth noting that previous exposure to similar problems may help (the basics of schema-based learning)
Also worth noting that there is perhaps a critical part of the configuration (the head) and noticing this may help the solver.

Part 1

What’s the problem?

What is Mathematics, really?

Methods

Formulas

Theorems

Patterns

Proofs

Rules

Definitions

Axioms

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What is mathematics, really? Methods, Formulas, Theorems, Patterns, Proofs, Rules, Definitions, Axioms? All these are essential but there is more to Mathematics.

What is Mathematics, really?

Mathematics is a playground where novices and experts alike can engage with problems and through this engagement, develop behaviours, habits of mind and an understanding of oneself that can be transformative.

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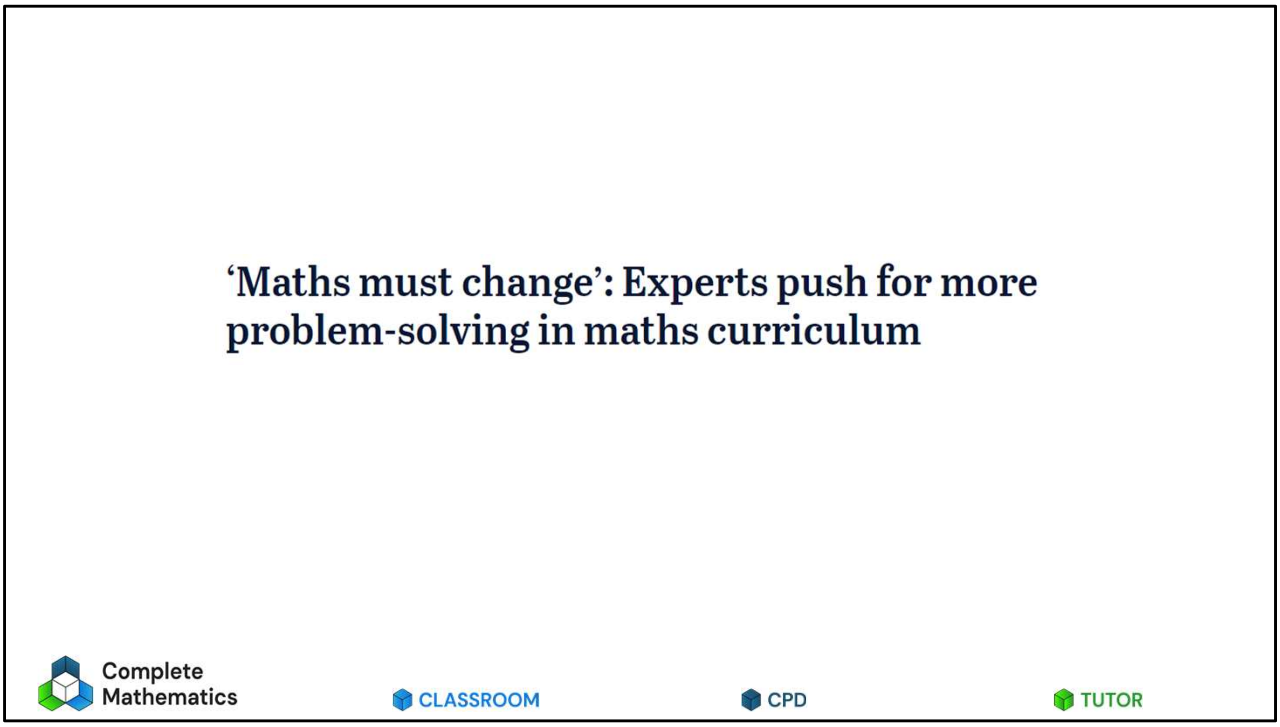
Mathematics is a playground where novices and experts alike can engage with problems large and small and through this engagement, develop behaviours, habits of mind and an understanding of oneself that can be transformative.



This perennial message has been restated countless times over decades in education reports, curriculum guidance, newspapers and can be heard coming from universities and employers in nearly every westernised country in the world.
Despite calls like these, we have to ask ourselves… Are pupils getting better at solving problems?

I'm going to say "no". In pockets, here and there, perhaps, but we're not seeing a general upwards trajectory in pupils' abilities to solve problems.

I would go as far as to argue that to compensate for poor problem solving, we have reduced many of the so-called problems in exams to little more than predictable, multi-step questions wrapped up in a little "context" that can be solved by any pupil who has seen, practised and remembered a similar question in the past.

There's been a great deal of research and implementation work done recently regarding instruction in mathematics education. In particular, we have a better understanding of how novices learn mathematics and how explicit instruction can be used to help pupils build knowledge and knowledge structures (schema). We have an understanding of purposeful and deliberate practice and we are starting to use what we know about memory to help long-term retention. Interleaving, interweaving and retrieval practice are becoming common phrases when teachers discuss effective pedagogies in mathematics.

However, despite all of this. We are still find pupils coming away from assessments saying: "I didn’t get that problem”, “I made a real mess of it”, “I didn’t know where to go next”, “I just didn’t know where to start”.

This is especially frustrating for us as we know the “problem” they are talking about is often not really a problem at all. Just some process they have – more often than not - succeeded with in that past.

Some definitions

Question – Any task assigned to a pupil

Exercise – A question solved by routine procedure

Problem – A question where the solution path is unclear

Some definitions: A question is a very general term. I am going to define it here as simply any task assigned to a pupil.

An exercise is a type of question where the solution requires only routine procedures. Provided they know the techniques, pupils simply follow an algorithm until they reach the solution. Exercises are important to provide practise and allow pupils to achieve fluency in procedures and techniques.

So, what is a problem: A problem is not an exercise but rather a question where the path to the solution is unclear. In other words, a problem arises when we are faced with a situation and are aware of a goal but do not know how to get from the initial state to the goal state.

Why solve problems?

The value gained by pupils when solving problems lies in how the pupil themselves becomes changed by the experience.

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By solving problems, pupils get to apply their knowledge in new ways. It gives opportunities to see connections and develop a deeper understanding of mathematical ideas.
Pupils get to experience being “stuck” and the satisfaction of finding a way to become “unstuck”. Essentially, they get a little taste of what it is like to be a mathematician and to behave mathematically.
Success is the best motivator and when they make a breakthrough in a problem, pupils will feel satisfaction and build an appetite for more.
Pupils confidence and self-belief grow and importantly, they see their hard work gets rewarded.
If they get the opportunity to reflect on their experiences then they will start to build up a bank of strategies and approaches that may help them with future problems.

What does it mean to solve a problem?

Not about getting the “answer”

More about devising a suitable strategy

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I’m going to suggest that one way of defining when a problem is solved is not at the point where the solver “gets the answer” – although this is obviously important - but at the point when the solver defines a valid “roadmap” - a mental representation of a solution path - that will take them from the initial state to the goal state.

In this way, solving problems is not so much about completing calculations (this obviously still matters) but about devising a suitable strategy.

What makes a good problem?

A good problem is one that mixes order and chaos in a deep and subtle way, and that fires the imagination for that reason.
Perhaps another way to say that is that in solving a good problem, one discovers some wonderful and totally unexpected regularity, when one expected nothing and on first sight saw only a jumble.

Douglas Hofstadter
From *Games and Mathematics* by David Wells

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A good problem should be pedagogically worthwhile

It should be relevant to the curriculum and should advance pupils knowledge and deepen their understanding

It should be matched to their ability, challenging while still being achievable in a reasonable amount of time

What makes a good problem?

- Contain some aspect(s) of novelty
- Bring together several mathematical ideas from different contexts
- Illustrate the application of one or more general principles that have wider application
- Lead somewhere by being capable of prompting new, related questions, suggesting generalisations or transformations

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What makes a good problem?

- Be capable of reformulation or translation into an equivalent problem with a known solution
- Form part of a sequence of questions leading to a significant piece of mathematical knowledge
- Call for some creative insight or flash of inspiration
- Have several different solutions, possibly varying in their simplicity or elegance

The ~~handshake~~ elbow bump problem

Mark is presenting a workshop at #mathsconf 30 delegates.

He asks all the delegates to “elbow bump” and introduce themselves to each other.

How many “elbow bumps” take place?

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This is a great problem as it meets many of the criteria from the previous slides.
It is accessible to younger pupils and can be attacked in different ways.
All through it’s solution, order comes from chaos and ultimately a general principle is revealed that has many wider applications.

The problem continuum: From park to wilderness

Image

Image

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Parks are pleasant places. The paths are easy to follow and in are good condition, there are rarely obstacles in the way. Usually you can see quite far ahead and although you can take a wrong turn, you are never far form a familiar place and will easily get back on track.

The wilderness is unpredictable, you sometimes have to forge your own path. There are obstacles and barriers that force you to change route. Sometimes you lose your sense of direction and sight of your destination.

Problems in mathematics are like this. Some are like a walk in the par whilst others are like an adventure in the wilderness.

Park problems are clearly defined, structured and easy to interpret. There are rarely any “wrong turns” and solution can be reduced to a linear sequence of straight-forward techniques.

Wilderness are unstructured and can be difficult to interpret. The goal state may be unknown. The solution will be nonlinear and may even lead to further problems. Solving a wilderness problem often requires insight; a sudden moment of clarity which moves the solution forwards. Insight is often thought to be “magical” but is more often the result of much thought and hard work.

A walk in the park for one…

Image

Image

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However, experience tells us that some pupils will have more success with a given problem than others. Why is this? What is different about these pupils?

In fact, a problem that is relatively straight-forward and is solved quickly by one pupil may be unfathomable to another.

In some cases, where the problem sits on the park-wilderness continuum is as much about the solver as the problem itself

In other words, a walk in the park for one solver may be a walk in the wilderness for another.

Jumping Frogs

Pink frogs can only move right. Blue can only move left.

Frogs can move forwards one space, or move two spaces by jumping over another frog.

Find the minimum number of moves needed until all frogs have switched positions.

https://nrich.maths.org/1246

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This is a example of a wilderness problem which meets many of the criteria discussed earlier.
It can be simplified by starting with 2 x 2 frogs
Generalisations are reached by considering n x n frogs
New problems emerge by considering a different starting number of pink to blue.

The mean height of a squad of 19 rugby players is 180 cm.

A player of height 174 cm joins the squad.

Find the new mean height of the squad.

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This is nearer to the "walk in the park" end of the continuum. It is typical of problems we see in textbooks and exams.
I think we still have to class it as a problem but it's a world away from the problems we've seen so far.

Some solvers will make short work of this, however, it is important to note that not all will find this a walk in the park. In fact, some solvers will be quite lost with this problem.

There are two reasons for this:
Firstly, this problem may not be appropriate for the solver's current level of mathematics – give this to a primary pupil who has never heard of "mean" and it is unlikely they will have much success.
Secondly, the solver may have met "mean" and might even have successfully solved similar problems in the past but has either forgotten some key knowledge or perhaps they never really developed enough understanding to allow them to solve it now.

As teachers, we can take two things from this…

Firstly, it is important to give pupils the right problems at the right time and secondly, one of the very best things we can do for pupils is to help them accrue knowledge and create situations where they have opportunities to develop conceptual understanding.

Being stuck

Mathematicians spend most of their time being stuck

Becoming unstuck requires us to behave mathematically

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Pupils need to experience what it feels like to be stuck.

One barrier to successful problem solving is emotional. Many pupils don't like not knowing what to do.
Think how often a pupil has asked you "is this right?" when their work is fine. Or, "Do I do this next?".

We want them to see being stuck as an opportunity. They should to experience struggle so they can solve problems independently and construct their own meaning.

However, there will be little gained by just throwing any old problem at a novice problem solver and expecting them to make meaning, form a plan, make decisions. This will likely have the opposite effect with pupils reaffirming their beliefs that problems are hard and they aren't bright enough to solve them.

Our role is to create situations where they can be stuck and find resolutions on their own without feeling rushed or inadequate. It's not practical to try to show them how to solve every problem there is but there are some things that we can do to help them become more mathematical.

Part 2

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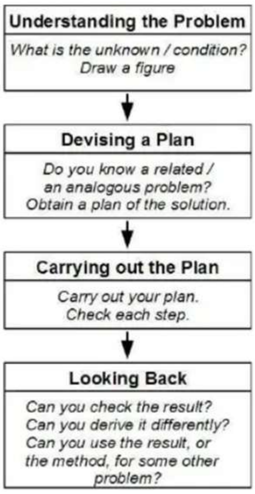
In part 1, I talked about why problem solving matters. We explored the features of a good problem, and I claimed that solving a problem was not so much about finding the "answer" but finding the way to the answer. I introduced the park/wilderness metaphor for describing problems and talked about why the difficulty in a problem is as much about the solver as it is the problem itself.

In part 2, I'll give my answer to the question "can we teach problem solving?". I'll give a brief history of research on problem solving, introduce some common problem-solving frameworks and discuss the differences in approaches taken by experts and novices when solving problems.

Polya



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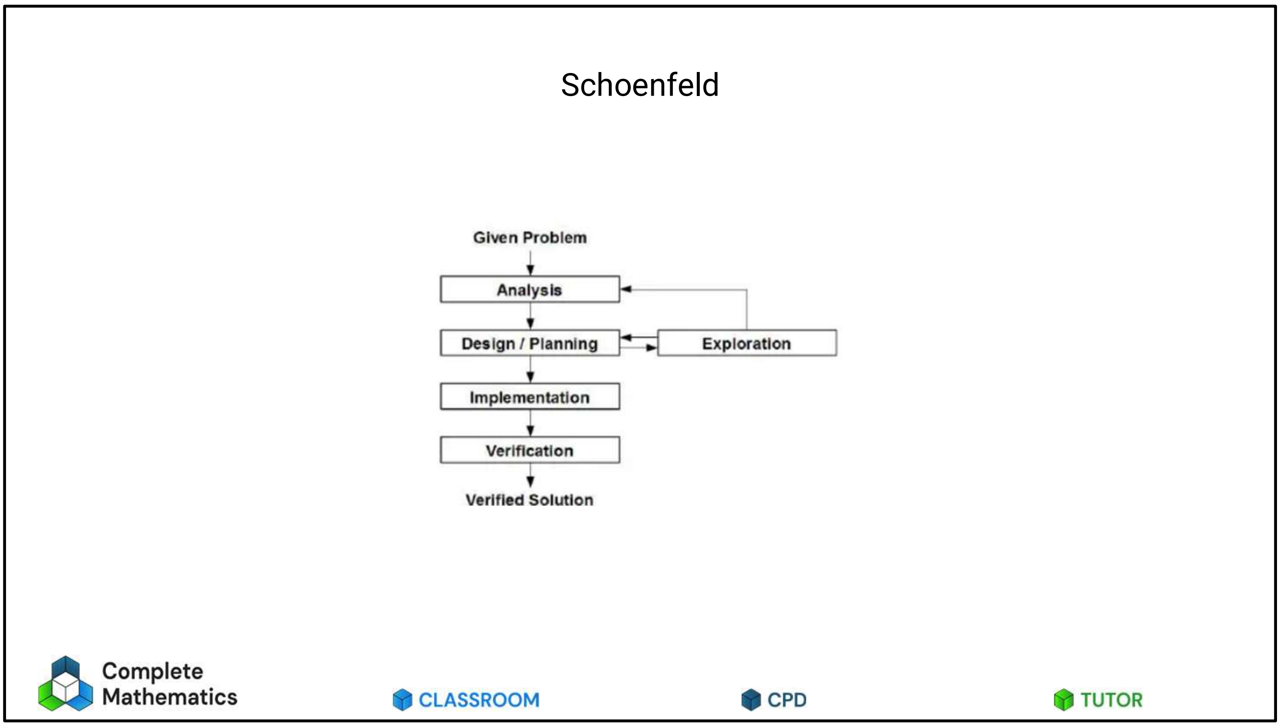
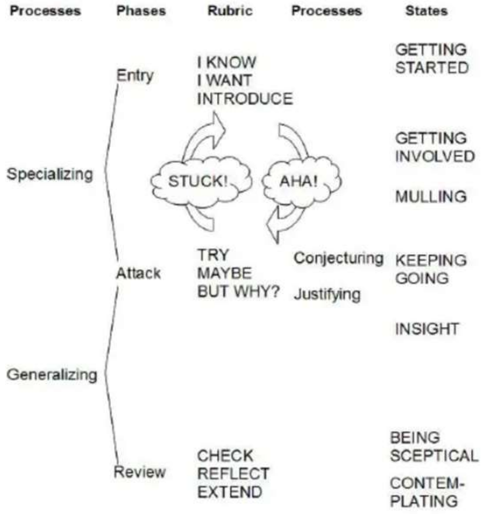
Georges Polya (1945)



Alan Schoenfeld (1985)

Breaks Polya’s linearity and introduces a feedback stage

Mason et al.



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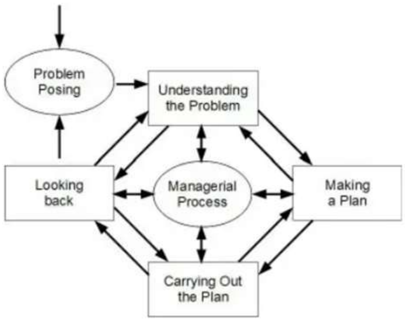
John Mason and colleagues (1982)

Mixed up Polya's steps but similarities can be seen – perhaps the most mathematics specific framework

Wilson et al.



James Wilson and colleagues (1994)

Introduces the managerial control centre and shows the non-linear nature of the problem solving process

A problem-solving framework

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Discuss and agree on a departmental framework to be used consistently with pupils. Perhaps use these as a guide.
Think about problem solving throughout the curriculum. It is quite possible to develop problem solving from an early stage so choose a framework that will hold "all the way through"

Can we teach problem solving?

Can we help pupils to get better at solving problems?

The McMaster Problem Solving Program

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Agreeing a framework and sharing it with pupils is a positive step but we are still faced with the question…

Can we teach problem solving?

The answer… well… no, and yes.

No, we can't teach problem solving in the same way that we teach trigonometry or calculus. Trigonometry and Calculus require the accumulation of relational knowledge but problem solving goes beyond knowledge, it is a way of thinking that requires the solver to use their knowledge in ways that are novel to them.
We shouldn't view problem solving as a stand-alone skill.
A 25 year project at McMaster University showed that giving students open-ended problems to solve didn't help as they get little feedback about the process steps and tended to rely on matching worked solutions to previous problems. Teachers working through lots of problems was also ineffective as teachers tend to "exercise solve". They tend to work forwards and don't make mistakes or struggle to form a representation of the problem. Even having students work through problems on the board so that different approaches can be seen didn't hep with students saying they became more confused and learned little.

A better question might be, can we help pupils to get better at solving problems? The answer to this is yes, we can.

Part 3

From the park to the wilderness

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The remainder of this presentation will look briefly at some strategies teachers can use to help pupils become better problem solvers.

Problems should feature at all stages of a pupils mathematical experiences – starting even before school.

In much the same way as our immune system needs exposure to small amounts of challenge to allow is to learn and become stronger over time, pupils should be gradually exposed to different problem types to enable them to develop the abilities that will allow them to successfully tackle similar problems in the future.

Good teaching matters

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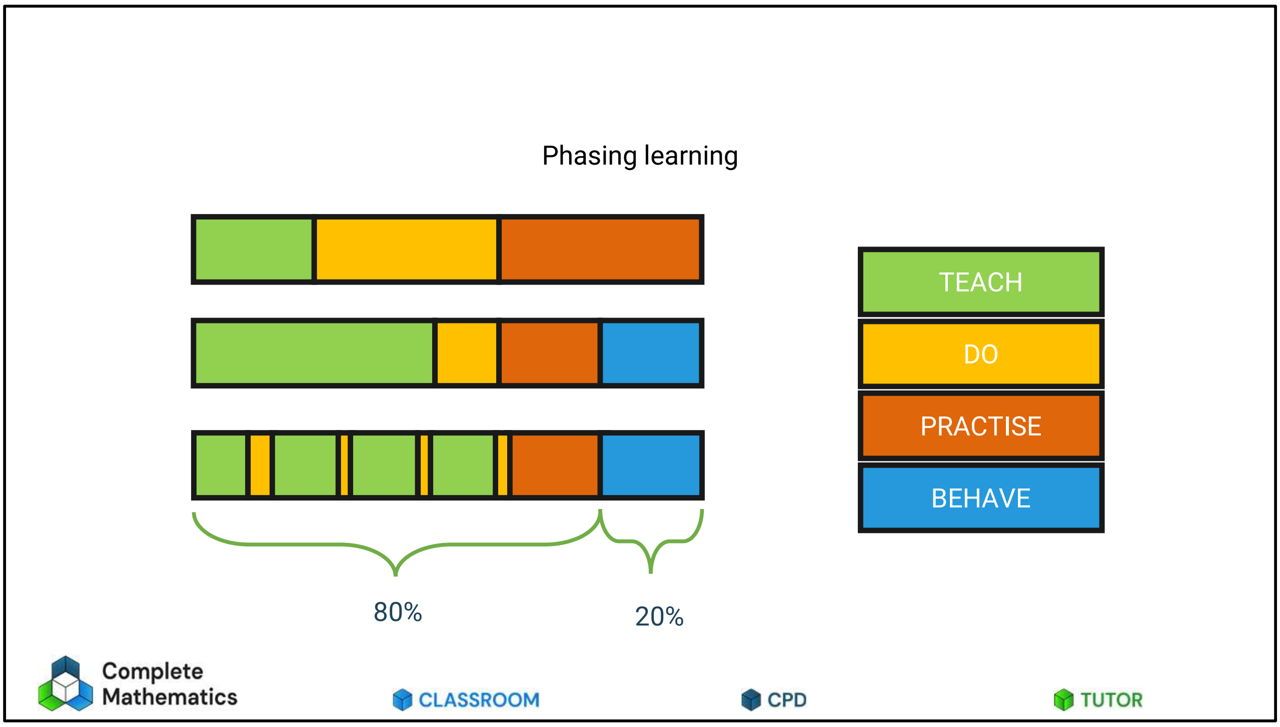
One of the very best things we can do for our pupils is to help each one of them build deep and interconnected knowledge structures. The more they know and understand, the better resourced they will be when it comes to solving problems. Gaps in knowledge or weak understanding present immediate and often insurmountable barriers to successful problem solving.

The next slide hints at it but I am not going to go into detail here by what I mean by good teaching – that is for another workshop - suffice to say that without it our pupils will continue to struggle when faced with problem-solving situations



These are the 4 phases we believe should be present in every mathematics learning episode in order for pupils to make the progress they are capable of.
Note, a learning episode is not necessarily (and often isn't) a single lesson. I will save a full explanation of these phases of leaning for another time and instead just focus today on the "Behave" phase.

Behave: Arguably the most important phase where pupils are encourages to behave mathematically. The important point here is that we are not asking them to "behave mathematically" with the novel idea from today's lesson. Too often, pupils can solve problems involving Pythagoras because all they have done that lesson is Pythagoras. We want to make the problem solving experience as authentic as possible. To do this we chose problems that require knowledge and understanding of an idea (or ideas) studied in the past. Sometimes quite far in the past.
This takes advantage of the "Retrieval" and "Spacing" effects to boost retention. Plus, pupils will see that they are now more confident in the earlier ideas which were (at some stage) novel and unfamiliar to them.

Other benefits of using ideas that have undergone a period of "maturation" are that…
Pupils struggling with today's novel idea are not excluded from today's problems – often the case when only the fast finishers get the "problems" at the end of exercises
And… teachers can plan the behave phase in advance and not risk having to change their plans in response to the lesson. In fact, Interleaving problems from earlier topics is not hard. The pupils will still have tackled the same questions – just not in the conventional order.

In this 20% behave time, the majority of problems will likely be nearer the “park” end of the continuum but time should also be set aside for a few forays into the wilderness.

Guided walks

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To ease the transition from park to wilderness, it is helpful to have a guide.
At first, the pupils will follow the guide into the wilderness, the guide will point out clearings, dead ends, explain navigational decision and even (at times) admit to being lost.
After time, the guide will accompany but not necessarily lead the pupils all the time – perhaps only offering suggestions at dead ends or cross-roads.
Eventually, the pupils will venture into the wilderness unaccompanied.

Of course, I'm really talking about teacher modelling and the gradual removal of guidance as pupils become more and more independent when solving problems.

If we want pupils to behave mathematically then they really should see us doing the same.

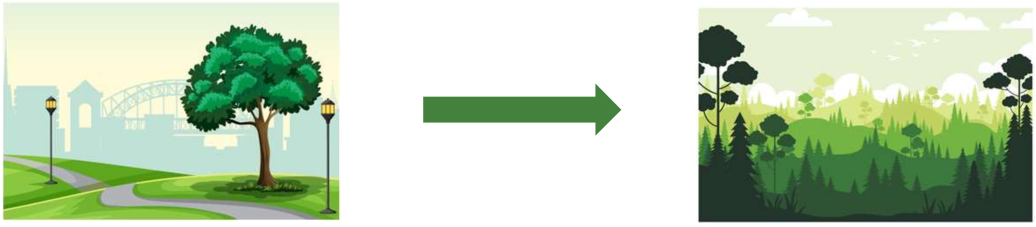
However, when being mathematical in front of pupils, we must remember that we are experts and pupils are novices. Just as a tourist experiences the wilderness in a different way to a local guide, what a pupil sees when presented with the solution to a problem may be very different to what the teacher sees…

Note that it can be difficult for an expert (i.e. the teacher) to recognise that a novice (the pupil) is experiencing something very different.
The teacher may be experiencing an instance of a general property, while students most often experience a particular.
The familiarity that comes with expertise tends to increase speed of talk and decrease verbal and physical pointing, reducing the opportunity for pupils to understand what is being said.

Narrate out loud your inner (novice) monologue. We want to demonstrate and model the dispositions that come with behaving mathematically. We want them to see us being stuck, trying things, rejecting ideas, trying alternatives, testing ideas and making breakthroughs. We really want pupils to see that the whole process is not frustrating, but quite good fun. That being challenged is an opportunity to try and learn new things.

During teacher modelling, pupils should be encouraged to identify when the teacher demonstrates good problem-solving behaviours, for example, using appropriate vocabulary, drawing a diagram, examine a simpler case, catching a mistake, reviewing the strategy, maintain a positive attitude, check work, etc.

Unguided walks



Robbins (2011) Problem Solving, Reasoning, and Analytical Thinking in a Classroom Environment

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Teacher modelling combined with “thinking aloud” can lead to peer modelling leading to individual work - emphasis on appropriate vocabulary; “explain it again better”

One idea might be to give pupils two problems and ask them to think about a suitable strategy for each and how they might put that into words. Then, the pupils pair up and pupil A explains their strategy for one of the problems to pupil B. All the time, pupil B is looking for examples of good problem solving behaviours. After some feedback, pupils B then explains their approach to the other problem. This task is not about getting the answer, but about thinking in detail about the solution path.

Ultimately, pupils are left to work independently on problems but are encouraged to continue the “think aloud” process which they have been practising. Of course, after time and for your sanity, you might want to have them use their “inner voice”!

Surface and deep structure

Willingham (2007) Critical Thinking Why Is It So Hard to Teach?

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Pupils thinking often focusses on the surface structure of a problem and they struggle to see the deeper structure of a problem. In one experiment, subjects were given 4 contextualised problems and detailed worked solutions and told to study then rate them in terms of clarity of writing. The subjects were then given a 5th problem, identical in deep structure to one of the earlier one only set in a different context, and asked to solve it. Fewer than 20% noticed that the new problem was similar to the previous one. It is thought that when we read a problem in context, our minds start digging up information related to the context, be it marching bands or vegetable patches and our thinking actually narrows and we are more likely to see the problems as different, rather than the same.

It takes a lot of successful practice with a problem type before pupils will know it well enough to recognise its deep structure but once pupils become very familiar with a problem's deep-structure, knowledge about how to solve it transfers well. That familiarity can come from long-term, repeated experience with one problem, or with various manifestations of one type of problem (i.e., many problems that have different surface structures, but the same deep structure). After repeated exposure to either or both, the subject simply perceives the deep structure as part of the problem description.

SSDD or DSSD

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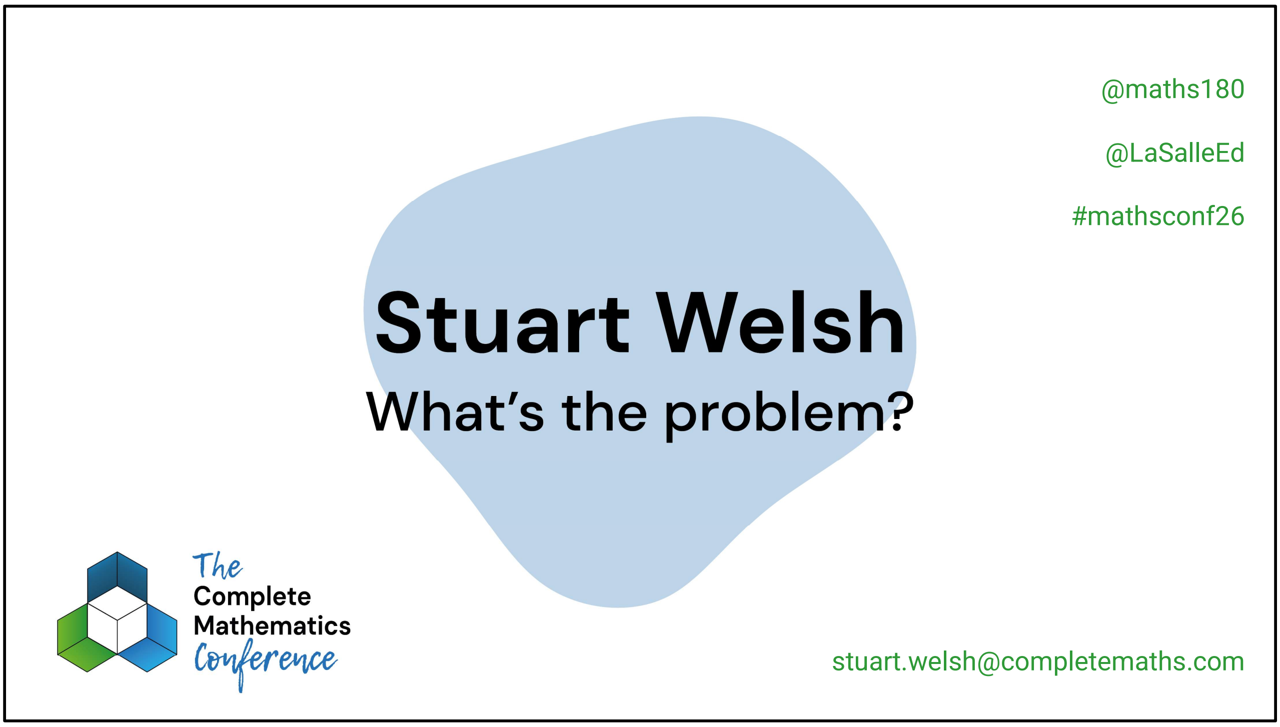
Same surface, different deep structure or Different surface, same deep structure

Consider offering pupils sets of problems that fit into either (or both) of these frameworks. These problems will generally be nearer the park end of the spectrum and could be given to pupils during the “Behave” phase of a learning episode.



So there we are. In part 3 we looked briefly at some strategies teachers can use to help pupils become better problem solvers.
We talked about why good teaching matters, the phasing of a learning episode, teacher modelling and the expert-novice distinction, talking aloud and about the surface and deep structure of problems.

I do hope you have found something of interest in this presentation. As I said at the start, this presentation is really an overview of a series of CPD sessions.
If you enjoyed this presentation, and think your department would enjoy some bespoke problem solving CPD, then get in touch on email or twitter.

Two passenger-less, run-away trains are heading towards each other, each travelling at a constant speed of 50 km/hour.
At the point when the trains are 200 km apart, a fly takes off from one train, flying straight above the rails at 75 km/hour to the other train, before bouncing off it and flying back to the first train.

This is repeated till the trains crash together and the fly is squashed!

What is the total distance covered by the fly before its tragic end?

There are two common ways of representing this problem…

Some will focus on the fly and set about calculating the sum of the ever-decreasing distances it travels from train to train.

Others will focus on the trains and set out to find the time it takes for them to collide, before working out how far the fly travels in this time.

Why is this?

The question asks for distance (travelled by the fly) which primes the solver to think about distances travelled by the fly.
It takes a flash of inspiration to see that it's far simpler to think first about the trains.

The trains meet after when each has travelled 100 km
This happens after 2 hours.
In that time, the fly has travelled 2 x 75 = 150 km.